

On the Complexity and Decidability of Some Problems Involving Shuffle[☆]

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Abstract

The complexity and decidability of various decision problems involving the shuffle operation (denoted by \sqcup) are studied. The following three problems are all shown to be NP-complete: given a nondeterministic finite automaton (NFA) M , and two words u and v , is $L(M) \not\subseteq u \sqcup v$, is $u \sqcup v \not\subseteq L(M)$, and is $L(M) \neq u \sqcup v$? It is also shown that there is a polynomial-time algorithm to determine, for NFAs M_1, M_2 and a deterministic pushdown automaton M_3 , whether $L(M_1) \sqcup L(M_2) \subseteq L(M_3)$. The same is true when M_1, M_2, M_3 are one-way nondeterministic l -reversal-bounded k -counter machines, with M_3 being deterministic. Other decidability and complexity results are presented for testing whether given languages L_1, L_2 and R from various languages families satisfy $L_1 \sqcup L_2 \subseteq R$, and $R \subseteq L_1 \sqcup L_2$. Several closure results on shuffle are also shown.

Keywords: Automata and Logic, Shuffle, Counter Machines, Pushdown Machines, Reversal-Bounds, Determinism, Commutativity, Strings

1. Introduction

The shuffle operator models the natural interleaving between strings. It was introduced by Ginsburg and Spanier [1], where it was shown that context-free languages are closed under shuffle with regular languages, but not context-free languages. It has since been applied in a number of areas such as concurrency [2], coding theory [3], verification [4], database schema [5], and biocomputing [3, 6], and has also received considerable study in the area of formal languages. However, there remains a number of open questions, such as the long-standing problem as to whether it is decidable, given a regular language R to tell if R has a non-trivial decomposition; that is, $R = L_1 \sqcup L_2$, for some L_1, L_2 that are not the language consisting of only the empty word [7].

This paper addresses several complexity-theoretic and decidability questions involving shuffle. In the past, similar questions have been studied by Ogden, Riddle, and Round [2], who showed that there exists deterministic context-free languages L_1, L_2 where $L_1 \sqcup L_2$ is NP-complete. More recently, L. Kari studied problems involving solutions to language equations of the form $R = L_1 \sqcup L_2$, where some of R, L_1, L_2 are given, and the goal is to determine a procedure, or determine that none exists, to solve for the variable(s) [8]. Also, there has been similar decidability problems investigated involving shuffle on trajectories [9], where

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the patterns of interleaving are restricting according to another language $T \subseteq \{0,1\}^*$ (a zero indicates that a letter from the first operand will be chosen next, and a one indicates a letter from the second operand is chosen). L. Kari and Sósík show that it is decidable, given L_1, L_2, R as regular languages with a regular trajectory set T , whether $R = L_1 \sqcup_T L_2$ (the shuffle of L_1 and L_2 with trajectory set T). Furthermore, if L_1 is allowed to be context-free, then the problem becomes undecidable as long as, for every $n \in \mathbb{N}$, there is some word of T with more than n 0's (with a symmetric result if there is a context-free language on the right). This implies that it is undecidable whether $L_1 \sqcup L_2 = R$, where R and one of L_1, L_2 are regular, and the other is context-free. In [10], it is demonstrated that given two linear context-free languages, it is not semi-decidable whether their shuffle is linear context-free, and given two deterministic context-free languages, it is not semi-decidable whether their shuffle is deterministic context-free. Complexity questions involving so-called *shuffle languages*, which are augmented from regular expressions by shuffle and iterated shuffle, have also been studied [11]. It has also been determined that it is NP-hard to determine if a given string is the shuffle of two identical strings [12].

Recently, there have been several papers involving the shuffle of two words. It was shown that the shuffle of two words with at least two letters has a unique decomposition into the shuffle of words [13]. In fact, the shuffle of two words, each with at least two letters, has a unique decomposition over arbitrary sets of words [14]. Also, a polynomial-time algorithm has been developed that, given a deterministic finite automaton (DFA) M and two words u, v , can test if $u \sqcup v \subseteq L(M)$ [15]. In the same work, an algorithm was presented that takes a DFA M as input and outputs a “candidate solution” u, v ; this means, if $L(M)$ has a decomposition into the shuffle of two words, u and v must be those two unique words. But the algorithm cannot guarantee that $L(M)$ has a decomposition. This algorithm runs in $O(|u| + |v|)$ time, which is often far less than the size of the input DFA, as DFAs accepting the shuffle of two words can be exponentially larger than the words [16]. It has also been shown [17] that the following problem is NP-complete: given a DFA M and two words u, v , is it true that $L(M) \not\subseteq u \sqcup v$?

In this paper, problems are investigated involving three given languages R, L_1, L_2 , and the goal is to determine decidability and complexity of testing if $R \not\subseteq L_1 \sqcup L_2$, $L_1 \sqcup L_2 \not\subseteq R$, and $L_1 \sqcup L_2 \neq R$, depending on the language families of L_1, L_2 and R . In Section 3, it is demonstrated that the following three problems are NP-complete: to determine, given an NFA M and two words u, v whether $u \sqcup v \not\subseteq L(M)$ is true, $L(M) \not\subseteq u \sqcup v$ is true, and $u \sqcup v \neq L(M)$ is true. Then, the DFA algorithm from [15] that can output a “candidate solution” is extended to an algorithm on NFAs that operates in polynomial time, and outputs two words u, v such that if the NFA is decomposable into the shuffle of words, then $u \sqcup v$ is the unique solution. And in Section 4, decidability and the complexity of testing if $L_1 \sqcup L_2 \subseteq R$ is investigated involving more general language families. In particular, it is shown that it is decidable in polynomial time, given NFAs M_1, M_2 and a deterministic pushdown automaton M_3 , whether $L(M_1) \sqcup L(M_2) \subseteq L(M_3)$. The same is true given M_1, M_2 that are one-way nondeterministic l -reversal-bounded k -counter machines, and M_3 , a one-way deterministic l -reversal-bounded k -counter machine. However, if M_3 is a nondeterministic 1-counter machine that makes only one reversal on the counter, and M_1 and M_2 are fixed DFAs accepting a^* and b^* respectively, then the question is undecidable. Also, if we have fixed languages $L_1 = (a + b)^*$ and $L_2 = \{\lambda\}$, and M_3 is an NFA, then testing whether $L_1 \sqcup L_2 \not\subseteq L(M_3)$ is PSPACE-complete. Also, testing whether $a^* \sqcup \{\lambda\} \not\subseteq L$ is NP-complete for L accepted by an NFA. For finite languages L_1, L_2 , and L_3 accepted by an NPDA, it is NP-complete to determine if $L_1 \sqcup L_2 \not\subseteq L_3$. Results on unary languages are also provided. In Section 5, testing $R \subseteq L_1 \sqcup L_2$ is addressed. This is already undecidable if R and L_1 are deterministic pushdown automata. However, it is decidable if L_1, L_2 are any commutative, semilinear languages, and R is a context-free language (even if augmented by reversal-bounded counters). Then, in Section 6, several other decision problems, and some closure properties of shuffle are investigated.

2. Preliminaries

We assume an introductory background in formal language theory and automata [18], as well as computational complexity [19]. We assume knowledge of pushdown automata, finite automata, and Turing machines, and we use notation from [18]. Let $\Sigma = \{a_1, \dots, a_m\}$ be a finite alphabet. Then Σ^* (Σ^+) is the set of all (non-empty) words over Σ . A language over Σ is any $L \subseteq \Sigma^*$. Given a language $L \subseteq \Sigma^*$, the

complement of L , $\bar{L} = \Sigma^* - L$. The length of a word $w \in \Sigma^*$ is $|w|$, and for $a \in \Sigma$, $|w|_a$ is the number of a 's in w .

Let \mathbb{N} be the positive integers, and \mathbb{N}_0 be the non-negative integers. For $n \in \mathbb{N}_0$, then define $\pi(n)$ to be 0 if $n = 0$, and 1 otherwise.

Next, we formally define reversal-bounded counter machines [20]. A *one-way k -counter machine* is a tuple $M = (k, Q, \Sigma, \triangleleft, \delta, q_0, F)$, where $Q, \Sigma, \triangleleft, q_0, F$ are respectively, the finite set of states, input alphabet, right input end-marker (not in Σ), the initial state, and the set of final states. The transition function δ is a mapping from $Q \times (\Sigma \cup \{\triangleleft\}) \times \{0, 1\}^k$ into $Q \times \{S, R\} \times \{-1, 0, +1\}^k$, such that if $\delta(q, a, c_1, \dots, c_k)$ contains (p, d, d_1, \dots, d_k) and $c_i = 0$ for some i , then $d_i \geq 0$ (this is to prevent negative values in any counter). The symbols S and R give the direction of the input tape head, being either *stay* or *right* respectively. Furthermore, M is deterministic if δ is a function. A configuration of M is a tuple (q, w, c_1, \dots, c_k) indicating that M is in state q with w (in Σ^* or $\Sigma^* \triangleleft$) as the remaining input, and $c_1, \dots, c_k \in \mathbb{N}_0$ are the contents of the counters. The derivation relation \vdash_M is defined by, $(q, aw, c_1, \dots, c_k) \vdash_M (p, w', c_1 + d_1, \dots, c_k + d_k)$, if $(p, d, d_1, \dots, d_k) \in \delta(q, a, \pi(c_1), \dots, \pi(c_k))$ where $d = S$ implies $w' = aw$, and $d = R$ implies $w' = w$. Then \vdash_M^* is the reflexive, transitive closure of \vdash_M . A word $w \in \Sigma^*$ is accepted by M if $(q_0, w \triangleleft, 0, \dots, 0) \vdash_M^* (q, \triangleleft, c_1, \dots, c_k)$, for some $q \in F, c_1, \dots, c_k \in \mathbb{N}_0$. The language accepted by M , $L(M)$, is the set of all words accepted by M . Essentially, a k -counter machine is a k -pushdown machine where each pushdown has one symbol plus an end-marker. It is well known that a two counter machine is equivalent to a deterministic Turing machine [21].

In this paper, we will often restrict the counter(s) to be reversal-bounded in the sense that each counter can only reverse (i.e., change mode from non-decreasing to non-increasing and vice-versa) at most r times for some given r . In particular, when $r = 1$, the counter reverses only once, i.e., once it decrements, it can no longer increment. Note that a counter that makes r reversals can be simulated by $\lceil \frac{r+1}{2} \rceil$ 1-reversal-bounded counters. Closure and decidable properties of various machines augmented with reversal-bounded counters have been studied in the literature (see, e.g., [20, 22, 23, 24]). We will use the notation $\text{NCM}(k, r)$ to represent r -reversal-bounded, k -counter machines, and NCM to represent all reversal-bounded multicounter machines. Machines with one unrestricted pushdown, plus reversal-bounded counters have also been studied [20]. Then, $\text{NPCM}(k, r)$ are machines with one unrestricted pushdown, and k r -reversal-bounded counters, and NPCM are all such machines. We use 'D' in place of 'N' for the deterministic versions, e.g., DCM , $\text{DCM}(k, r)$, DPCM , and $\text{DPCM}(k, r)$. We use this notation for both the classes of machines, and the families of languages they accept.

We will also use the notation below to represent common classes of automata (and languages): NPDA for nondeterministic pushdown automata; DPDA for deterministic pushdown automata; NCA for nondeterministic one counter machines with no reversal-bound; DCA for deterministic NCAs; NFA for nondeterministic finite automata; DFA for deterministic finite automata; and DTM for deterministic Turing machines. As is well-known, NFAs, NPDAs, halting DTMs, and DTMs, accept exactly the regular languages, context-free languages, recursive languages, and recursively enumerable languages, respectively.

A set $Q \subseteq \mathbb{N}_0^m$ is a *linear set* if there exist vectors $\vec{v}_0, \vec{v}_1, \dots, \vec{v}_n \in \mathbb{N}_0^m$ such that $Q = \{\vec{v}_0 + i_1 \vec{v}_1 + \dots + i_n \vec{v}_n \mid i_1, \dots, i_n \in \mathbb{N}_0\}$. In this definition, the vector \vec{v}_0 is called the constant and $\vec{v}_1, \dots, \vec{v}_n$ are the *periods*. A *semilinear set* is a finite union of linear sets. For semilinear sets $Q_1, Q_2 \subseteq \mathbb{N}^m$, $Q_1 + Q_2 = \{v \mid v_1 + v_2, v_1 \in Q_1, v_2 \in Q_2\}$.

The *Parikh map* of $w \in \Sigma^*$, $\Sigma = \{a_1, \dots, a_m\}$ is the vector $\psi(w) = (|w|_{a_1}, \dots, |w|_{a_m})$, and the Parikh map of L is $\psi(L) = \{\psi(w) \mid w \in L\}$. For a vector $\vec{v} \in \mathbb{N}_0^m$, the inverse $\psi^{-1}(\vec{v}) = \{w \in \Sigma^* \mid \psi(w) = \vec{v}\}$, which is extended to subsets of \mathbb{N}_0^m . A language is *semilinear* if its Parikh map is semilinear. The commutative closure of a language $L \subseteq \Sigma^*$ is $\text{comm}(L) = \psi^{-1}(\psi(L))$, and a language L is *commutative* if it is equal to its commutative closure. The family of all commutative semilinear languages is denoted by COM-SLIP , following notation developed in [25].

Let $u, v \in \Sigma^*$. The *shuffle* of u and v , denoted $u \sqcup v$ is the set

$$\{u_1 v_1 u_2 v_2 \dots u_n v_n \mid u_i, v_i \in \Sigma^*, 1 \leq i \leq n, u = u_1 \dots u_n, v = v_1 \dots v_n\}.$$

This can be extended to languages $L_1, L_2 \subseteq \Sigma^*$ by $L_1 \sqcup L_2 = \bigcup_{u \in L_1, v \in L_2} u \sqcup v$. Given $u, v \in \Sigma^*$, there is an obvious NFA with $(|u| + 1)(|v| + 1)$ states accepting $u \sqcup v$, where each state stores a position within both

u and v . This has been called the *naive NFA* for $u \sqcup v$ [16]. It was also mentioned in [16] that if u and v are over disjoint alphabets, then the naive NFA is a DFA.

An NFA $M = (Q, \Sigma, q_0, F, \delta)$ is *accessible* if, for each $q \in Q$, there exists $u \in \Sigma^*$ such that $q \in \delta(q_0, u)$. Also, M is *co-accessible* if, for each $q \in Q$, there exists $u \in \Sigma^*$ such that $\delta(q, u) \cap F \neq \emptyset$. Lastly, M is *trim* if it is both accessible and co-accessible, and M is *acyclic* if $q \notin \delta(q, u)$ for every $q \in Q, u \in \Sigma^+$.

3. Comparing Shuffle on Words to NFAs

The results to follow in this section depend on the following result from [17].

Proposition 1. *It is NP-complete to determine, given a DFA M and words u, v over an alphabet of at least two letters, if $L(M) \not\subseteq u \sqcup v$.*

First, we note that this NP-completeness can be extended to NFAs.

Corollary 2. *It is NP-complete to determine, given an NFA M and words u, v over an alphabet of at least two letters, if $L(M) \not\subseteq u \sqcup v$.*

PROOF. NP-hardness follows from Proposition 1.

To show it is in NP, let M be an NFA with state set Q . Create a nondeterministic Turing machine that guesses a word w of length at most $|uv| + |Q|$, and verify that $w \in L(M)$ and that $w \notin u \sqcup v$ in polynomial time [17]. And indeed, $L(M) \not\subseteq u \sqcup v$ if and only if $L(M) \cap \{w \mid |w| \leq |uv| + |Q|, w \in \Sigma^*\} \not\subseteq u \sqcup v$, since any word longer than $|uv| + |Q|$ that is in $L(M)$ implies there is another one in $L(M)$ with length between $|uv| + 1$ and $|uv| + |Q|$, which is therefore not in $u \sqcup v$ (all words in $u \sqcup v$ are of length $|u| + |v|$). \square

Next, the reverse inclusion of Corollary 2 will be examined. In contrast to the polynomial-time testability of $u \sqcup v \subseteq L(M)$ when M is a DFA ([15], with an alternate shorter proof appearing in Proposition 8 of this paper), testing $u \sqcup v \not\subseteq L(M)$ is NP-complete for NFAs.

Proposition 3. *It is NP-complete to determine, given an NFA M and u, v over an alphabet of at least two letters, whether $u \sqcup v \not\subseteq L(M)$.*

PROOF. First, it is in NP, since all words in $u \sqcup v$ are of length $|uv|$, and so a nondeterministic Turing machine can be built that nondeterministically guesses one and tests if it is not in $L(M)$ in polynomial time.

For NP-hardness, let F be an instance of the 3SAT problem (a known NP-complete problem [19]) with a set of Boolean variables $X = \{x_1, \dots, x_p\}$, and a set of clauses $\{c_1, \dots, c_q\}$, where each clause has three literals.

If d is a truth assignment, then d is a function from X to $\{+, -\}$ (true or false). For a variable x , then x^+ and x^- are literals. In particular, the literal x^+ is true under d if and only if the variable x is true under d . And, the literal x^- is true under d if and only if the variable x is false [19]. Let $y = \lceil \log_2 p \rceil + 1$, which is enough to hold the binary representation of any of $1, \dots, p$. For an integer i , $1 \leq i \leq p$, let $b(i)$ be the string 1 followed by the y -bit binary representation of i , followed by 1 again.

For $1 \leq i \leq p, 1 \leq j \leq q$, let $f(i, j)$ be defined as follows, where each element is a set of strings over $\{0, 1\}$:

$$f(i, j) = \begin{cases} \{01b(i)\}, & \text{if } x_i^+ \in c_j; \\ \{10b(i)\}, & \text{if } x_i^- \in c_j; \\ \{10b(i), 01b(i)\}, & \text{otherwise.} \end{cases}$$

For $1 \leq j \leq q$, let $F_j = f(1, j)f(2, j) \cdots f(p, j)$.

We will next give the construction. Let $u = 1b(1)1b(2) \cdots 1b(p)$, and let $v = 0^p$.

Let $T = \{e_1b(1)e_2b(2) \cdots e_pb(p) \mid e_i \in \{10, 01\}, 1 \leq i \leq p\}$. Clearly $T \subseteq u \sqcup v$, and also T is a regular language, and a DFA M_T can be built accepting this language in polynomial time, as with a DFA $\overline{M_T}$ accepting $\overline{L(M_T)}$.

Then, make an NFA M' accepting $\bigcup_{1 \leq j \leq q} F_j$. It is clear that this NFA is of polynomial size. Then, make another NFA M'' accepting $L(M') \cup L(\overline{M_T})$. The following claim shows that deciding $u \sqcup v \not\subseteq L(M'')$ is equivalent to deciding if there is a solution to the 3SAT instance.

Claim 1. *The following three conditions are equivalent:*

1. $u \sqcup v \cap \overline{L(M'')} \neq \emptyset$,
2. $T \cap \overline{L(M'')} \neq \emptyset$,
3. F has a solution.

PROOF. “1 \Rightarrow 2”. Let $w \in u \sqcup v \cap \overline{L(M'')}$. Then $w \notin L(M'')$, and since $L(\overline{M_T}) \subseteq L(M'')$, necessarily $w \in T$.

“2 \Rightarrow 1”. Let $w \in T \cap \overline{L(M'')}$. But, $T \subseteq u \sqcup v$; and so $w \in u \sqcup v \cap \overline{L(M'')}$.

“2 \Rightarrow 3”. Assume $w \in T \cap \overline{L(M'')}$. Thus, $w = e_1b(1)e_2b(2) \cdots e_pb(p)$, $e_i \in \{10, 01\}$, but $w \notin \bigcup_{1 \leq j \leq q} F_j$. Let d be the truth assignment obtained from w where

$$d(x_i) = \begin{cases} +, & \text{if } e_i = 10; \\ -, & \text{if } e_i = 01; \end{cases}$$

for all $i, 1 \leq i \leq p$. Thus, for every $j, 1 \leq j \leq q$, $w \notin F_j$, but for all variables x_i not in c_j , $e_ib(i)$ must be an infix of words in F_j since $10b(i)$ and $01b(i)$ are both in $f(i, j)$ when x_i is not in c_j . So one of the words encoding the (three) variables in c_j , must have $10b(i)$ as an infix of words in F_j where $d(x_i) = +$, or $01b(i)$ as an infix of words in F_j where $d(x_i) = -$, since otherwise F_j would have as infix, for each x_i that is a variable of c_j , $01b(i)$ if $x_i^+ \in c_j$, and $10b(i)$ if $x_i^- \in c_j$, and so w would be in F_j , a contradiction. Thus, d makes clause c_j true, as is the case with every clause. Hence, d is a satisfying truth assignment, and F is satisfiable.

“3 \Rightarrow 2”. Assume F is satisfiable, hence d is a satisfying truth assignment. Create

$$w = e_1b(1)e_2b(2) \cdots e_pb(p),$$

where

$$e_i = \begin{cases} 10, & \text{if } d(x_i) = +; \\ 01, & \text{if } d(x_i) = -. \end{cases}$$

Then $w \in T$. Also, for each j , d applied to some variable, say x_i , must be in c_j , but then by the construction of F_j , $e_ib(i)$ must not be an infix of any word in F_j . Hence, $w \notin \bigcup_{1 \leq j \leq q} F_j$, $w \notin L(M')$, and $w \notin L(M'')$. Hence, $w \in T \cap \overline{L(M'')}$. □

□

Next, we examine the complexity of testing inequality between languages accepted by NFAs and words of a very simple form.

Proposition 4. *It is NP-complete to test, given $a^p, b^q \in \Sigma^*$, $p, q \in \mathbb{N}_0$, and M an NFA over $\Sigma = \{a, b\}$, whether $L(M) \neq a^p \sqcup b^q$.*

PROOF. First, it is immediate that the problem is in NP, by Corollary 2 and Proposition 3.

To show NP-hardness, the problem in Proposition 1 is used.

Given M , a DFA, and words u, v , we can construct the naive shuffle NFA N for $u \sqcup v$. The naive NFA is of polynomial size in the length of u and v . Let $(p, q) = (|uv|_a, |uv|_b)$. Then construct the naive NFA A accepting $a^p \sqcup b^q$, which is a polynomially sized DFA since a^p, b^q are over disjoint alphabets. Thus, another DFA can be built accepting $\overline{L(A)}$. We can then construct an NFA M' in polynomial time which accepts

$(a^p \sqcup b^q \cap \overline{L(M)}) \cup L(N) \cup (L(M) \cap \overline{(a^p \sqcup b^q)})$ as M is already a DFA. Also, $u \sqcup v \subseteq a^p \sqcup b^q$ since the latter contains all words with p a 's and q b 's.

We will show $L(M) \subseteq u \sqcup v$ if and only if $L(M') = a^p \sqcup b^q$.

Assume $L(M) \subseteq L(N) (= u \sqcup v)$. Then $L(M) \cap \overline{(a^p \sqcup b^q)} = \emptyset$ since $u \sqcup v \subseteq a^p \sqcup b^q$. All other words in $L(M')$ are in $a^p \sqcup b^q$. Thus, $L(M') \subseteq a^p \sqcup b^q$. Let $w \in a^p \sqcup b^q$. If $w \notin L(M)$, then $w \in L(M')$. If $w \in L(M)$, then $w \in L(N) \subseteq L(M')$, by the assumption.

Assume $L(M') = a^p \sqcup b^q$. Let $w \in L(M)$. Then $L(M) \cap \overline{(a^p \sqcup b^q)} = \emptyset$ by the assumption. So, $L(M) \subseteq a^p \sqcup b^q$. Assume $w \in L(M)$ but $w \notin L(N)$. However, $w \in L(M')$ by the assumption, a contradiction, as $w \notin a^p \sqcup b^q \cap \overline{L(M)}$, and $w \notin L(M) \cap \overline{a^p \sqcup b^q}$, implying $w \in L(N)$.

Hence, the problem is NP-complete. \square

To obtain the following corollary, it only needs to be shown that the problem is in NP, which again follows from Corollary 2 and Proposition 3.

Corollary 5. *It is NP-complete to determine, given an NFA M and words u, v over an alphabet of size at least two, if $L(M) \neq u \sqcup v$.*

It is known that there is a polynomial-time algorithm that, given a DFA, will output two words u and v such that, if $L(M)$ is decomposable into the shuffle of two words, then this implies $L(M) = u \sqcup v$ [15]. Moreover, this algorithm runs in time $O(|u| + |v|)$, which is sublinear in the input M . This main result from [15] is as follows:

Proposition 6. *Let M be an acyclic, trim, non-unary DFA over Σ . Then we can determine words $u, v \in \Sigma^+$ such that, $L(M)$ has a shuffle decomposition into two words implies $L(M) = u \sqcup v$ is the unique decomposition. This can be calculated in $O(|u| + |v|)$ time.*

However, the downside to this algorithm is that it can output two strings u and v , when $L(M)$ is not decomposable. Thus, the algorithm does not check whether $L(M)$ is decomposable, but if it is, it can find the decomposition in time usually far less than the number of states of the DFA. The decomposition also must be unique over words (this is always true when there are at least two combined letters) [13].

It is now shown that this result scales to NFAs, while remaining polynomial time complexity. The algorithm in [15] scans at most $O(|u| + |v|)$ transitions and states of the DFA from initial state towards final state. From an NFA, it becomes possible to apply the standard subset construction [18] on the NFA only by creating states and transitions for the transitions and states examined by this algorithm (thus, the NFA is never fully determinized, and only a subset of the transitions and states of the DFA are created and traversed). Because the algorithm essentially follows one “main” path from initial state to final state in the DFA, the amount of work required for NFAs is still polynomial.

Proposition 7. *There is a polynomial-time complexity algorithm that, given an acyclic, non-unary NFA $M = (Q_N, \Sigma, q_{N0}, F_N, \delta_N)$, can find strings $u, v \in \Sigma^+$, such that, $L(M)$ has a decomposition into two words implies $L(M) = u \sqcup v$ is the unique decomposition. Moreover, this algorithm runs in time $O((|u| + |v|) |Q_N|^2)$.*

Proof sketch. Uniqueness again follows from [13].

The algorithm outputs words $u, v \in \Sigma^+$ such that either $L(M) = u \sqcup v$ or M is not shuffle decomposable. It is based off the one described in [15], which is quite detailed, and thus not reproduced here, although we will refer to it.

In order to use the algorithm in Proposition 6, first all states that are not accessible or not co-accessible are removed. For this, a breadth-first graph search algorithm can be used to detect which states can be reached from q_0 in $O(|Q_N|^2)$ time. It also verifies that all final states reached are the same distance from the initial state, and if not, M is not decomposable. Then, collapse these final states down to one state q_f and remove all outgoing transitions, which does not change the language accepted since M is acyclic. Then, check which states can be reached from q_f following transitions in reverse using the graph search, and

remove all states that cannot be reached. This results in an NFA $M_1 = (Q_1, \Sigma, q_1, \{q_f\}, \delta_1)$ that is trim and accepts $L(M)$.

Let $M_D = (Q_D, \Sigma, q_{D0}, F_D, \delta_D)$ be the DFA obtained from M_1 via the subset construction (we do not compute this, but will refer to it). Necessarily M_D is trim and acyclic, since M_1 is as well. Then $q_{D0} = \{q_1\}$, $F_D = \{P \mid P \in Q_D, q_f \in P\}$.

We modify the algorithm of Proposition 6 as follows: In place of DFA states, we use subsets of Q_1 from Q_D [18]. However, states and transitions are only computed as needed in the algorithm. Any time $\delta(P, a)$ is referenced in the algorithm, we first compute the deterministic transition as follows: $\delta_D(P, a) = \bigcup_{p \in P} \delta_1(p, a)$, and then use this transition. Since there are at most $|Q_1|$ states in a subset of Q_1 , any transition of δ_D defined on a given state and a given letter can transition to at most $|Q_1|$ states. Then, we can compute $\delta_D(P, a)$ in $O(|Q_1|^2)$ time (for each state $p \in P$, add $\delta_1(p, a)$ into a sorted list without duplicates). As it is making the list, it can test if this state is final by testing if $q_f \in \delta_D(P, a)$. Therefore, this algorithm inspects $O(|u| + |v|)$ states and transitions of M_D , which takes $O(|Q_1|^2(|u| + |v|))$ time to compute using the subset construction. \square

4. Testing Inclusion of the Shuffle of Languages in Another Language

A known result involving shuffle on words is that there is a polynomial-time test to determine, given words $u, v \in \Sigma^+$ and a DFA M , whether $u \sqcup v \subseteq L(M)$ [15]. An alternate simpler proof of this result is demonstrated next, and then this proof technique will be used to extend to more general decision problems.

Proposition 8. *There is a polynomial-time algorithm to determine, given $u, v \in \Sigma^+$, and a DFA M , whether or not $u \sqcup v \subseteq L(M)$.*

PROOF. Clearly, $u \sqcup v$ is a subset of $L(M)$ if and only if $L(A) \cap \overline{L(M)} = \emptyset$, where A is the naive NFA accepting $u \sqcup v$. A DFA accepting $\overline{L(M)}$ can be built in polynomial time, and the intersection is accepted by an NFA using the standard construction [18] whose emptiness can be checked in polynomial time [18]. \square

This result will be generalized in two ways. First, instead of individual words u and v , languages from $\text{NCM}(k, r)$, for some fixed k, r will be used. Moreover, instead of a DFA for the right side of the inclusion, a $\text{DCM}(k, r)$ machine will be used.

Proposition 9. *Let k, r be any fixed integers. It is decidable, given $M_1, M_2 \in \text{NCM}(k, r)$ and $M_3 \in \text{DCM}(k, r)$, whether $L(M_1) \sqcup L(M_2) \subseteq L(M_3)$. Moreover, the decision procedure is polynomial in $n_1 + n_2 + n_3$, where n_i is the size of M_i .*

PROOF. First, construct from M_1 and M_2 , an NCM M_4 that accepts $L(M_1) \sqcup L(M_2)$. Clearly M_4 is an $\text{NCM}(2k, r)$, and the size of M_4 is polynomial in $n_1 + n_2$.

Then, construct from M_3 a $\text{DCM}(k, r)$ machine M_5 accepting the complement of $L(M_3)$, which can be done in polynomial time [20].

Lastly, construct from M_4 and M_5 an $\text{NCM}(3k, r)$ machine M_6 accepting $L(M_4) \cap L(M_5)$ by simulating the machines in parallel.

It is immediate that $L(M_1) \sqcup L(M_2) \subseteq L(M_3)$ if and only if $L(M_6) = \emptyset$. Further, it has been shown that for any fixed t, s , it is decidable in polynomial time, given M in $\text{NCM}(t, s)$, whether $L(M) = \emptyset$ [26]. \square

Actually, the above proposition can be made stronger. For any fixed k, r , the decidability of non-emptiness of $L(M)$ for an $\text{NCM}(k, r)$ is in NLOG , the class of languages accepted by nondeterministic Turing machines in logarithmic space [26]. It is known that NLOG is contained in the class of languages accepted by deterministic Turing machines in polynomial time (whether or not the containment is proper is open). By careful analysis of the constructions in the proof of the above proposition, M_6 , could be constructed by a logarithmic space deterministic Turing machine. Hence:

Corollary 10. *Let k, r be any fixed integers. The problem of deciding, given $M_1, M_2 \in \text{NCM}(k, r)$ and $M_3 \in \text{DCM}(k, r)$, whether $L(M_1) \sqcup L(M_2) \subseteq L(M_3)$, is in NLOG .*

Proposition 9 also holds if M_1 and M_2 are NFAs and M_3 is a deterministic pushdown automaton.

Proposition 11. *It is decidable, given NFAs M_1, M_2 and $M_3 \in \text{DPDA}$, whether $L(M_1) \sqcup L(M_2) \subseteq L(M_3)$. Moreover, the decision procedure is polynomial in $n_1 + n_2 + n_3$, where n_i is the size of M_i .*

PROOF. The proof and algorithm proceeds much like the proof of Proposition 9. Given two NFAs M_1, M_2 , another NFA M_4 that accepts $L(M_1) \sqcup L(M_2)$ can be constructed in polynomial time. Then, for a given DPDA M_3 , a DPDA M_5 can be constructed accepting its complement in polynomial time (and is of polynomial size) [27]. Also, given an NFA M_4 and a DPDA, a NPDA M_6 can be built in polynomial time accepting $L(M_4) \cap L(M_5)$. As above, $L(M_1) \sqcup L(M_2) \subseteq L(M_3)$ if and only if $L(M_6) = \emptyset$, and emptiness is decidable in polynomial time for NPDAs [18]. \square

In contrast to Proposition 11, the following is shown:

Proposition 12. *It is undecidable, given one-state DFAs M_1 accepting a^* and M_2 accepting b^* , and an NCM(1, 1) machine M_3 over $\{a, b\}$, whether $L(M_1) \sqcup L(M_2) \subseteq L(M_3)$.*

PROOF. Let $\Sigma = \{a, b\}$. Then $L_1 \sqcup L_2 = \Sigma^*$. Let $M_3 \subseteq \Sigma^*$ be an NCM(1, 1) machine. Then $L_1 \sqcup L_2 \subseteq L_3$ if and only if $L_3 = \Sigma^*$. The result follows, since the universality problem for NCM(1, 1) is undecidable [28]. The idea is the following: Given a single-tape deterministic Turing machine Z , we construct M_3 which, when given any input w , accepts if and only if w does not represent a halting sequence of configurations of Z on an initially blank tape (by guessing a configuration ID_i , and extracting the symbol at a nondeterministically chosen position j within this configuration, storing j in the counter, and then checking that the symbol in position j in the next configuration ID_{i+1} determined by decrementing the counter is not compatible with the next move of the DTM from ID_i ; see [28]). Hence, $L(M_3)$ accepts the universe if and only if Z does not halt. By appropriate coding, the universe can be reduced to $\{a, b\}^*$. \square

Note that the proof of Proposition 12 shows: Let G be a language family such that universality is undecidable. Then it is undecidable, given one-state DFAs M_1 accepting a^* and M_2 accepting b^* , and L in G , whether $L(M_1) \sqcup L(M_2) \subseteq L$.

Proposition 13. *Let $L_1 = (a + b)^*$ and $L_2 = \{\lambda\}$. It is PSPACE-complete, given an NFA M with input alphabet $\{a, b\}$, whether $L_1 \sqcup L_2 \not\subseteq L(M)$.*

PROOF. Clearly, $(a + b)^* \sqcup \{\lambda\} \not\subseteq L$ if and only if $L \neq (a + b)^*$. The result follows, since it is known that this question is PSPACE-complete (see, e.g., [19]). \square

Remark 2. In Proposition 12, if M_1 and M_2 are DFAs accepting finite languages, and L is a language in any family with a decidable membership problem, then it is decidable whether $L(M_1) \sqcup L(M_2) \subseteq L$. This is clearly true by enumerating all strings in $L(M_1) \sqcup L(M_2)$ and testing membership in L .

Next, shuffle over unary alphabets is considered.

Proposition 14. *It is decidable, given languages L_1, L_2, L_3 over alphabet $\{a\}$ accepted by NPCMs, whether $L_1 \sqcup L_2 \subseteq L_3$.*

PROOF. It is known that the Parikh map of the language accepted by any NPCM is an effectively computable semilinear set (in this case over \mathbb{N}) [20] and, hence, the languages L_1, L_2, L_3 can be accepted by DFAs over a unary alphabet. \square

Proposition 15. *It is NP-complete to decide, for an NFA M over alphabet $\{a\}$, whether $a^* \sqcup \{\lambda\} \not\subseteq L(M)$.*

PROOF. Clearly, $a^* \sqcup \{\lambda\} \not\subseteq L$ if and only if $L \neq a^*$. The result follows, since it is known that this question is NP-complete (see, e.g., [19]). \square

For the case when the unary languages L_1 and L_2 are finite:

Proposition 16. *It is polynomial-time decidable, given two finite unary languages L_1 and L_2 (where the lengths of the strings in L_1 and L_2 are represented in binary) and a unary language L_3 accepted by an NFA M , all over the same letter, whether $L_1 \sqcup L_2 \subseteq L_3$.*

PROOF. Let r be the sum of the cardinalities of L_1 and L_2 , s be the length of the binary representation of the longest string in $L_1 \cup L_2$, and t be the length of binary representation of M .

We represent the NFA M by an $n \times n$ Boolean matrix A_M , where n is the number of states of M , and $A_M(i, j) = 1$ if there is a transition from state i to state j ; 0 otherwise.

Let x be the binary representation of a unary string a^d , where $d = d_1 + d_2$, $a^{d_1} \in L_1$, and $a^{d_2} \in L_2$. To determine if a^d is in L_3 , we compute A_M^d and check that for some accepting state p , the $(1, p)$ entry is 1. Since the computation of A_M^d can be accomplished in $O(\log d)$ Boolean matrix multiplications (using the “right-to-left binary method for exponentiation” technique used to compute x^m where m is a positive integer in $O(\log m)$ multiplications, described in Section 4.6.3 of [29]), and since matrix multiplication can be calculated in polynomial time, it follows that we can decide whether $L_1 \sqcup L_2 \subseteq L_3$ in time polynomial in $r + s + t$. \square

However, when the alphabet of the finite languages L_1, L_2 is at least binary:

Proposition 17. *It is NP-complete to determine, given finite language L_1 and L_2 , and an NPDA M accepting L_3 , whether $L_1 \sqcup L_2 \subseteq L_3$.*

PROOF. NP-hardness follows from Proposition 3. To show that it is in NP, guess a word $u \in L_1$, and $v \in L_2$, guess a word w of length $|u| + |v|$, and verify that it is in $u \sqcup v$ [17]. Then, verify that $w \notin L_3$, which can be done in polynomial time since the membership problem for NPDAs can be solved in polynomial time. \square

Proposition 18. *It is undecidable, given languages L and L_1 accepted by 1-reversal-bounded DPDAs (resp., DCAs), whether $L_1 \sqcup \{\lambda\} \subseteq L$.*

PROOF. It is known that the disjointness problem for 1-reversal-bounded DPDAs (resp., DCAs) is undecidable (see, e.g., [20]). Let $L, L_1 \in \text{DPDA}$. Then $L \cap L_1 = \emptyset$ if and only if $L_1 \subseteq \overline{L}$ if and only if $L_1 \sqcup \{\lambda\} \subseteq \overline{L}$, and \overline{L} is in DPDA as it is closed under complement. \square

5. Testing Inclusion of a Language in the Shuffle of Languages

In this section, the reverse containment is addressed. That is, given L, L_1, L_2 , is $L \subseteq L_1 \sqcup L_2$? This question will depend on the language families where L, L_1 , and L_2 belong.

First, for 1-reversal-bounded DPDAs, the following is immediate. The proof is identical to that of Proposition 18.

Proposition 19. *It is undecidable, given languages L and L_1 accepted by 1-reversal-bounded DPDAs (resp., DCAs), whether $L \subseteq L_1 \sqcup \{\lambda\}$.*

The following proposition follows from the known undecidability of the universality problem for NCM(1, 1) [28].

Proposition 20. *If $L_1 \in \text{NCM}(1, 1)$, $L_2 = \{\lambda\}$, and L is the fixed regular language Σ^* , then it is undecidable whether $L \subseteq L_1 \sqcup L_2$.*

Remark 3: We are currently examining the question of whether it is undecidable, given a regular language L and languages L_1, L_2 accepted by DPDAs (resp., DCAs, DPCMs), whether $L \subseteq L_1 \sqcup L_2$. In particular, we are interested in the simple case when L and L_1 are regular and L_2 is in DCM(1, 1).

Next, we examine some families where decidability occurs. The following is true since the shuffle of regular languages is regular, and the containment problem is decidable for regular languages.

Proposition 21. *If L_1, L_2, L are all regular languages, then it is decidable whether $L \subseteq L_1 \sqcup L_2$.*

The following is true since the shuffle of an NPCM and an NCM is an NPCM language, and containment can be decided by using the decidable membership problem for NPCM [20].

Proposition 22. *If $L_1 \in \text{NPCM}$, $L_2 \in \text{NCM}$, and L is finite, then it is decidable whether $L \subseteq L_1 \sqcup L_2$.*

Proposition 23. *If $L \in \text{NPCM}$, and $L_1, L_2 \in \text{DCM}$ over disjoint alphabets, then it is decidable whether $L \subseteq L_1 \sqcup L_2$.*

PROOF. Let $M_1, M_2 \in \text{DCM}$ over disjoint alphabets, such that $L_1 = L(M_1)$ and $L_2 = L(M_2)$, where M_1 has k_1 counters and M_2 has k_2 counters. Then $L_1 \sqcup L_2$ can be accepted by a $k_1 + k_2$ counter machine (by simulating M_1 on the first k_1 counters and M_2 on the last k_2 counters). Furthermore, since L_1, L_2 are over disjoint alphabets, $L_1 \sqcup L_2$ is in DCM as well. Then we can construct $\overline{L_1 \sqcup L_2}$ since DCM is closed under complement [20], and test if $L \cap \overline{L_1 \sqcup L_2} = \emptyset$, which is true if and only if $L \subseteq L_1 \sqcup L_2$. \square

The following can also be shown with a proof identical to Proposition 14.

Proposition 24. *It is decidable, given languages L_1, L_2, L over alphabet $\{a\}$ accepted by NPCMs, whether: $L \subseteq L_1 \sqcup L_2$.*

Lastly, a large family is presented for which decidability follows. These questions will be examined next for commutative languages. First, two lemmas are needed.

Lemma 25. *Let $\Sigma = \{a_1, \dots, a_m\}$. We can effectively construct, given semilinear sets $Q_1, Q_2 \subseteq \mathbb{N}^m$, a semilinear set Q such that $Q = Q_1 + Q_2$. Further, we can construct a DCM M_Q to accept $\psi^{-1}(Q)$.*

PROOF. It is known that every COM-SLIP language is in DCM. Therefore, for $i = 1, 2$, a DCM M_{Q_i} can be constructed accepting $\psi^{-1}(Q_i)$.

Then construct an NCM M to accept $\{a_1^{k_1} \dots a_m^{k_m} \mid k_1 = r_1 + s_1, \dots, k_m = r_m + s_m, a_1^{r_1} \dots a_m^{r_m} \in \psi^{-1}(Q_1), a_1^{s_1} \dots a_m^{s_m} \in \psi^{-1}(Q_2)\}$ as follows: given input $w = a_1^{k_1} \dots a_m^{k_m}$, M reads the input and nondeterministically guesses the decompositions of the k_i 's into r_i 's and s_i 's, and stores them in $2m$ counters which we call $c_1, \dots, c_m, d_1, \dots, d_m$ (they store the numbers $r_1, \dots, r_m, s_1, \dots, s_m$). Then M simulates the computation of M_{Q_1} on input $a_1^{r_1} \dots a_m^{r_m}$ (by decrementing the counters c_1, \dots, c_m which have values r_1, \dots, r_m , corresponding to reading an input letter of input M_{Q_1}) and if M_{Q_1} accepts, M then simulates the computation of M_{Q_2} on input $a_1^{s_1} \dots a_m^{s_m}$ using counters d_1, \dots, d_m similarly. Then M accepts if M_{Q_2} accepts. Since NCM only accepts semilinear languages, it follows that there is a semilinear set Q such that $\psi(L(M)) = Q$, and hence $\psi^{-1}(Q)$ can be accepted by a DCM. \square

The next result follows from the definition of commutative semilinear languages and the previous lemma.

Lemma 26. *Let $Q_1, Q_2 \subseteq \mathbb{N}^m$ be semilinear sets, and $Q = Q_1 + Q_2$. Then $\psi^{-1}(Q) = \psi^{-1}(Q_1) \sqcup \psi^{-1}(Q_2)$. Moreover, Q can be effectively constructed from Q_1 and Q_2 .*

We can now prove:

Proposition 27. *Let $\Sigma = \{a_1, \dots, a_m\}$, $m \geq 1$. It is decidable, given an NPCM M accepting a language $L \subseteq \Sigma^*$, and COM-SLIP languages $L_1, L_2 \subseteq \Sigma^*$ (effectively semilinear), whether $L \subseteq L_1 \sqcup L_2$. Further, for $L, L_1, L_2 \in \text{NPCM}$, it is decidable whether $L \subseteq \text{comm}(L_1) \sqcup \text{comm}(L_2)$.*

PROOF. The second statement follows from the first since NPCM is effectively semilinear and both $\text{comm}(L_1)$ and $\text{comm}(L_2)$ are in COM-SLIP.

Using the semilinear sets $Q_1 = \psi(L_1)$, $Q_2 = \psi(L_2)$, then from Lemma 25 and Lemma 26, we can construct a DCM M_1 accepting $L_1 \sqcup L_2$. We can then construct a DCM M_2 accepting $\overline{L(M_1)}$, since DCM is closed under complementation. Then we construct an NPCM M_3 (simulating M and M_2 in parallel) accepting $L(M) \cap L(M_2)$. The result follows, since $L(M) \subseteq L_1 \sqcup L_2$ if and only if $L(M_3) = \emptyset$, which is decidable, since the emptiness problem for NPCMs is decidable [20]. \square

This result generalizes from NPCM to other effectively semilinear language families, such as the nondeterministic versions of the semilinear automata models from [30].

The reverse inclusion of the above proposition is not true however, since if $L_1 = (a + b)^*$ and $L_2 = \{\lambda\}$ (trivially commutative semilinear languages), then it is undecidable, given an NCM(1, 1) M with input alphabet $\{a, b\}$, whether $L_1 \sqcup L_2 \subseteq L(M)$. This is because it is undecidable whether an NCM(1, 1) machine is equal to $\{a, b\}^*$.

6. Other Decision and Closure Properties Involving Shuffle

The following proposition follows from the proof of Theorem 6 in [10].

Proposition 28. *It is undecidable, given two languages accepted by 1-reversal-bounded DPDAs (resp., DCAs), whether their shuffle is accepted by a 1-reversal-bounded DPDA (resp., DCA).*

We can show the undecidability of a related problem:

Proposition 29. *It is undecidable, given two 1-reversal-bounded DPDAs M_1, M_2 (resp., DCAs) and a 2-state DFA M , whether $L(M) \cap (L(M_1) \sqcup L(M_2)) = \emptyset$.*

PROOF. Let L_1 and L_2 be accepted by 1-reversal-bounded DPDAs (resp., DCAs) over input alphabet Σ . Let $\Sigma' = \{a' \mid a \in \Sigma\}$. Define the homomorphism h by: $h(a) = a'$ for all $a \in \Sigma$. Let $L_3 = \{w' \mid w' = h(w), w \in L_1\}$. (Thus, L_3 is a primed version of L_1 .) Let $L = \{a_1 a'_1 \cdots a_k a'_k \mid k \geq 0, a_1, \dots, a_k \in \Sigma\}$. Clearly, L is regular and can be accepted by a 2-state DFA. Now $L \cap (L_1 \sqcup L_3) = \emptyset$ if and only if $L_1 \cap L_2 = \emptyset$, which is undecidable. \square

Now, we consider the shuffle of languages from various families, and contrast closure results with established results on commutative languages.

Clearly, if L_1, L_2 are in NCM, then $L_1 \sqcup L_2$ is also in NCM. Thus, NCM is closed under shuffle. However, we have:

Proposition 30. *There are languages $L_1, L_2 \in \text{DCM}(1, 1)$ such that $L_1 \sqcup L_2$ is over a two letter alphabet, but not a context-free language (NPDA).*

PROOF. Let $L_1 = \{a^n bab^n a \mid n > 0\}$ and $L_2 = \{b^m a^{m+1} \mid m > 0\}$. If $L = L_1 \sqcup L_2$ is a context-free language, then $L' = L \cap \{a^i b^j a^k b^l a^p \mid i, l \geq 1, j, p > 1, k = 2\} = \{a^n b^{m+1} a^2 b^n a^{m+1} \mid n, m > 0\}$ is also a context-free language. But it is easy to show, using the Pumping Lemma, that L' is not a context-free language. \square

This contrasts the commutative case as $L_1 \sqcup L_2$ is over a two letter alphabet and is semilinear, but not a context-free language, but every two letter semilinear language that is commutative is a context-free language.

It is known that the commutative closure of every NPCM (or effectively semilinear language family) is a DCM language [31]. By contrast, the shuffle of two 1-reversal DPDAs can be significantly more complex, and might not even be an NPCM language, and can create non-semilinear languages.

Proposition 31. *There are languages L_1, L_2 accepted by 1-reversal-bounded DPDAs (resp., DCAs) such that $L_1 \sqcup L_2$ is not in NPCM.*

PROOF. For Part 1, let

$$\begin{aligned} L_1 &= \{a^{i_1} \# a^{i_3} \# a^{i_5} \# \cdots \# a^{i_{2n-1}} \# a^{i_{2n}} \# \cdots \# a^{i_6} \# a^{i_4} \# a^{i_2} \mid n \geq 3, i_1 = 1, i_{j+1} = \\ &\quad i_j + 1 \text{ for odd } j\}, \\ L_2 &= \{a^{i_1} \# a^{i_3} \# a^{i_5} \# \cdots \# a^{i_{2n-1}} \# a^{i_{2n}} \# \cdots \# a^{i_6} \# a^{i_4} \# a^{i_2} \mid n \geq 3, i_1 = 1, i_{j+1} = \\ &\quad i_j + 1 \text{ for even } j\}. \end{aligned}$$

Clearly, L_1, L_2 can be accepted by 1-reversal DPDAs. Let $L = L_1 \cap L_2$. Then the Parikh map of L is not semilinear since it has the same Parikh map as the language $\{a^1 \# a^2 \# a^3 \# \dots \# a^{2n} \mid n \geq 3\}$. Hence, L cannot be accepted by an NPCM, since it is known that the Parikh map of any NPCM language is semilinear [20].

Now let L_1, L_2 be over alphabet $\Sigma = \{a, \#, \$\}$. Let $\Sigma' = \{a', \#', \$'\}$ be the “primed” copy of Σ . For any $x \in \Sigma^*$, let x' be primed version of x (i.e., the symbols in x are replaced by their primed copies). Let $L'_2 = \{x' \mid x \in L_2\}$. Clearly, we can construct a 1-reversal-bounded DPDA accepting L'_2 .

Suppose $L_3 = L_1 \sqcup L'_2$ can be accepted by an NPCM. We can also then construct an NPCM accepting $L_4 = L_3 \cap \{ss' \mid s \in \Sigma\}^*$. Let h be a homomorphism that maps the primed symbols to λ (i.e., they are erased) and leaves the un-primed symbols unchanged. Since NPCM languages are closed under homomorphism, $h(L_4)$ can also be accepted by an NPCM. This gives a contradiction, since $h(L_4) = L = L_1 \cap L_2$ cannot be accepted by an NPCM.

The proof for Part 2 is similar, except that we modify the languages L_1, L_2 , as follows:

$$\begin{aligned} L_1 &= \{a^{i_1} \# a^{i_2} \# a^{i_3} \# a^{i_4} \# a^{i_5} \# a^{i_6} \# \dots \# a^{i_{2n-1}} \# a^{i_{2n}} \mid n \geq 3, i_1 = 1, \\ &\quad i_{j+1} = i_j + 1 \text{ for odd } j\}, \\ L_2 &= \{a^{i_1} \# a^{i_2} \# a^{i_3} \# a^{i_4} \# a^{i_5} \# a^{i_6} \# \dots \# a^{i_{2n-1}} \# a^{i_{2n}} \mid n \geq 3, i_1 = 1, \\ &\quad i_{j+1} = i_j + 1 \text{ for even } j\}. \end{aligned}$$

Clearly, L_1, L_2 can be accepted by DCAs. The rest of the proof is similar to that of Part 1. \square

If in the statement of Proposition 31, one of L_1, L_2 is accepted by an NCM, then the proposition is no longer true, since it can be easily verified that $L_1 \sqcup L_2$ could then be accepted by an NPCM. In contrast, for NCM, there are languages accepted by deterministic counter automata L_1 , such that $L_1 \sqcup \{\lambda\}$ is not in NCM.

We need the following lemma.

Lemma 32. $L = \{a^n b^n \mid n > 0\}$ is in DCM(1, 1), but L^+ (and, hence, also L^*) is not in NCM.

PROOF. It is obvious that L is in DCM(1, 1). Suppose L^+ can be accepted by an NCM M . Consider the following languages:

$$\begin{aligned} L_1 &= \{a^n b^{n+1} \mid n > 0\}^+ \{a^m \mid m > 0\}, \\ L_2 &= \{a^1\} \{b^n a^{n+1} \mid n > 0\}^+. \end{aligned}$$

Clearly, we can construct from M , NCMs M_1 and M_2 accepting L_1 and L_2 , respectively. Since the family of NCM languages is closed under intersection, $L_3 = L_1 \cap L_2 = \{a^1 b^2 a^3 b^4 \dots a^n b^{n+1} a^{n+2} \mid n > 0\}$ is also in NCM. The result follows, since the Parikh map of any NCM language is semilinear [20], but the Parikh map of L_3 is not semilinear. \square

Proposition 33. *There is a language L_1 (L^+ in Lemma 32) accepted by a DCA such that $L_1 \sqcup \{\lambda\}$ is not in NCM.*

Therefore, commutative closure creates much simpler languages than just the identity operation.

7. Conclusions and Open Problems

We investigated the complexity and decidability of various decision problems involving the shuffle operation. In particular, we showed that the following three problems are NP-complete for a given NFA M , and two words u and v :

- Is $L(M) \not\subseteq u \sqcup v$?
- Is $u \sqcup v \not\subseteq L(M)$?
- Is $L(M) \neq u \sqcup v$?

We showed that there is a polynomial-time algorithm to determine, for NFAs M_1, M_2 , and a deterministic pushdown automaton M_3 , whether $L(M_1) \sqcup L(M_2) \subseteq L(M_3)$. The same is true when M_1, M_2, M_3 are one-way nondeterministic l -reversal-bounded k -counter machines, with M_3 being deterministic.

We also presented decidability and complexity results for testing whether given languages L_1, L_2 and L from various languages families satisfy $L_1 \sqcup L_2 \subseteq L$.

We obtained some closure properties of the shuffle operation on languages. In particular, we proved:

1. There are languages $L_1, L_2 \in \text{DCM}(1, 1)$ such that $L_1 \sqcup L_2$ is over a two letter alphabet, but is not a context-free language (NPDA).
2. There are languages L_1, L_2 accepted by 1-reversal-bounded DPDAs (resp., DCAs) such that $L_1 \sqcup L_2$ is not in NPCM.
3. There is a language L_1 accepted by a DCA such that $L_1 \sqcup \{\lambda\}$ is not in NCM.

There are a number of remaining open problems including the following:

1. Is it undecidable, given a regular language L and languages L_1, L_2 accepted by DPDAs (resp., DCAs, DPCMs), whether $L \subseteq L_1 \sqcup L_2$? In particular, we are interested in the simple case when L and L_1 are regular and L_2 is in $\text{DCM}(1, 1)$.
2. Same as item (1) above except that now the question is whether $L = L_1 \sqcup L_2$.
3. What is the complexity of testing, given a DFA M , and words u, v , whether $L(M) \neq u \sqcup v$?

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